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New theories with quantum modified moduli space

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Abstract: Kutasov-type duals of supersymmetric gauge theories had been studied only in the dual regime and the s-confining case. Here we extend the discussion to the case of less flavor, analogous to the case of quantum-modified moduli space in Seiberg duality. Unlike the Seiberg duality, however, we find that parts of the moduli space become superconformal, generalizing the so far isolated example of SU(2) theory with two doublets and a triplet. We also point out that the magnetic superpotential needs to be augmented by an additional instanton-generated piece when the magnetic group is SU(2).

Keywords: Nonperturbative Effects, Solitons Monopoles and Instantons.

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Contents

1. Introduction

Many of recent advances on understanding gauge theories and string theories rely on the notion of duality. The concept of duality varies depending on the context. In the case of string dualities, they are supposed to be exact equivalence between two theories. Two theories are different descriptions of the same theory valid in different limits. In the case of field theory dualities, they mostly refer to the equivalence of low-energy limits, a generalization of the concept of universality class in critical phenomena. Supersymmetric gauge theories have proven to be wonderful testing grounds for various types of dualities.

Among dualities in supersymmetric gauge theories, Kutasov-type dualities [2, 3] offer the most complex and rich phenomena of duality. The original version by Kutasov-Schwimmer is a duality between SU(N) gauge theory with one adjoint chiral multiplet X and F quark favors, supplemented by a superpotential of the form $W = h\lambda \operatorname{Tr} X^{k+1}$ and an SU($kF - N$) theory with a similar particle content but more complicated superpotential. Furthermore, a later paper together with Seiberg [4] demonstrated a beautiful structure of chiral rings, discussed originally in the context of two-dimensional superconformal theories.

JHEP05(2002)055

This type of duality had later been extended to many other gauge groups and both symmetric and anti-symmetric tensor fields [5]. They also offer the most non-trivial examples of discrete anomaly matching [6].

Despite the successes mentioned above, Kutasov-type dualities still suffer from the lack of a complete picture. For instance, the dynamics in the absence of the superpotential is still a complete mystery. For smaller number of flavors, we expect analogues of $F \leq N+1$ cases of Seiberg's $\mathrm{SU}(N)$ dualities; however the only case that had been worked out in full detail is the case of $kF - N = 1$, an analogue of $F = N+1$ leading to new s-confining theories. Indeed, a free theory of composites with an irrelevant superpotential was found.

The purpose of this paper is to investigate the dynamics of the $\mathrm{SU}(N)$ Kutasov-type duals when $F = kN$. Naively, this gives an analogue of $F = N$ case in Seiberg duality with a quantum-modified moduli space. We show that such a picture is almost correct, but with an important difference. At certain sections of the quantum-modified moduli space, there appear interacting non-trivial superconformal theories. The closest example that exhibits a similar behavior studied in the literature so far is the $\mathrm{SU}(2)$ theory with two doublets and a triplet [10]. Therefore our work generalizes this isolated example to a whole class of theories.

The paper is organized as follows. In section 2, we review the known results from Kutasov-Schwimmer and later with Seiberg on $\mathrm{SU}(N)$ gauge theory with F flavors and an adjoint X. This result applies only when $kF - N > 1$. In section 3, we study the electric theory with $kF = N$. In section 4 we study the corresponding magnetic theory by decoupling a flavor from the $\mathrm{SU}(k)$ theory with $F+1$ flavors. We draw our conclusions in section 5. Additional checks of Kutasov duality are considered in the appendix. There we also discuss a subtlety of the case $k=2$, $2F - N = 2$: it turns out that the usual Kutasov superpotential for the magnetic theory is incomplete in that case, and an additional term must be added to reproduce the result of the electric theory.

2. Overview of the theory and its dual

2.1 Known Results

In this section, we review the known Kutasov-type duals. Consider a $\mathrm{SU}(N)$ gauge theory with F flavors in fundamental (Q) and anti-fundamental ($\overline{Q}$) representation and an adjoint (X), and with a superpotential

$$W = \frac{h}{k+1} \operatorname{Tr} X^{k+1}, \tag{2.1}$$

where h is a coupling constant of dimension $k-2$. The symmetry properties of the matter fields are taken to be as in table 1.

If we impose the D-flatness conditions and mod out by gauge transformations (or, equivalently, mod out by complexified gauge transformations), we can transform X to a

	$\mathrm{SU}(N)$	$\mathrm{SU}(F)$	$\mathrm{SU}(F)$	$\mathrm{U}(1)$	$\mathrm{U}(1)_R$
X	Adj	1	1	0	$\frac{2}{k+1}$
Q	$\square$	$\square$	1	1	$1-\frac{2N}{(k+1)F}$
$\overline{Q}$	$\overline{\square}$	1	$\overline{\square}$	-1	$1-\frac{2N}{(k+1)F}$

Table 1: ???

Jordan normal form

$$X = \begin{pmatrix} a & 1 & & & & & & & \\ & a & 1 & & & & & & \\ & & a & & & & & & \\ & & & b & 1 & & & & \\ & & & & b & & & & \\ & & & & & c & & & \\ & & & & & & \cdot & & \\ & & & & & & & \cdot & \\ & & & & & & & & z \end{pmatrix} . \tag{2.2}$$

The moduli space of the theory is then obtained by imposing the F-flatness condition from (2.1):

$$X^k - \frac{1}{N}\,\mathrm{Tr}\,X^k = 0\,. \tag{2.3}$$

This way, we end up with two possibilities: either X is diagonal and it is $X^k = v^k I$, where v is an arbitrary complex number and I is the identity matrix, or it has all zero diagonal entries (and thus it is a singular matrix with vanishing kth power and the D-term is canceled by corresponding contributions from quarks). We can think of the latter case as the origin of the flat direction v in moduli space.

For $v \neq 0$, vacua of the gauge theory can be labeled by sequences of integers $(r_1, r_2, \ldots, r_k)$, with $\sum r_i = N$, where r_i is the number of eigenvalues of X residing in the $i-$th of the k roots of v^k. The gauge group is broken by the X expectation value to

$$\mathrm{SU}(N) \to \mathrm{SU}(r_1) \times \mathrm{SU}(r_2) \times \cdots \times \mathrm{SU}(r_k) \times \mathrm{U}(1)^{k-1}\,. \tag{2.4}$$

Obviously, $\sum_{i=1}^k r_i = N$. Thus, at low energies we are left with k decoupled QCD theories with gauged baryon number(s). The quantum behavior of each $\mathrm{SU}(r_i)$ is known from the pioneering work by Seiberg [1]. In particular, remember that classical vacua with $r_i \leq F$ are removed from the quantum moduli space (so, at least for $v \neq 0$, theories with the matter content of table 1 can have stable vacua at all only if $kF \geq N$). For $r_i > F$ the quantum moduli space is identical to the classical one, and for $r_i = F$ the two spaces are different, the classical (compositeness) relation between mesons and baryons, $\det M^{(i)} - B^{(i)}\overline{B}^{(i)} = 0$, being replaced by the quantum one:

$$\det M^{(i)} - B^{(i)}\overline{B}^{(i)} = \Lambda^{(i)3r_i - F}\,. \tag{2.5}$$

JHEP05(2002)055

	$\mathrm{SU}(kF-N)$	$\mathrm{SU}(F)$	$\mathrm{SU}(F)$	$\mathrm{U}(1)$	$\mathrm{U}(1)_R$
Y	Adj	1	1	0	$\frac{2}{(k+1)}$
q	$\square$	$\overline{\square}$	1	$\frac{N}{kF-N}$	$1-\frac{2(kF-N)}{(k+1)F}$
$\overline{q}$	$\overline{\square}$	1	$\square$	$-\frac{N}{kF-N}$	$1-\frac{2(kF-N)}{(k+1)F}$
M_j	1	$\square$	$\overline{\square}$	0	$2-\frac{4N}{(k+1)F}+\frac{2}{(k+1)}(j-1)$

Table 2: ???

Moreover, remember that there are strong pieces of evidence [1] that a $\mathrm{SU}(r)$ theory with F flavors is dual to a $\mathrm{SU}(F-r)$ theory with F flavors, one singlet and an appropriate tree-level superpotential. As a natural generalization of this duality conjecture, refs [2, 3, 4] suggested that the theory with the field content of table 1 is dual to a theory with $\mathrm{SU}(kF-r)$ gauge group, F flavors of (dual) quarks (q) and antiquarks ($\overline{q}$), k singlets M_j and an adjoint Y, with the symmetry properties as in table 2.

Except for the case $k=2$, $2F-N=2$, the magnetic tree-level superpotential is taken to be

$$W_{\mathrm{magn}} = -\frac{h}{k+1}\,\mathrm{Tr}\,Y^{k+1} + \frac{h}{\mu^2}\sum_i M_i \overline{q} Y^{k-i} q\,. \tag{2.6}$$

The F-flatness condition for Y is similar in form to that of eq. (2.3), therefore the magnetic moduli space also contains a flat direction v, analogous to that of the electric theory. Points in the electrical moduli space where the $\mathrm{SU}(N)$ theory splits into the product of the k $\mathrm{SU}(r_i)$ theories correspond to points in the magnetic moduli space where the magnetic $\mathrm{SU}(kF-N)$ theory splits into the product of the corresponding dual $\mathrm{SU}(F-r_i)$ theories. Notice that points in the classical electric moduli space with some $r_i < F$ are removed from the corresponding magnetic moduli space because $\mathrm{SU}(F-r_i)$ cannot exist then. Thus, the two spaces do not agree classically, but only quantum mechanically.

In the case $k=2$, $2F-N=2$, $\mathrm{Tr}\,Y^3=0$, and, as shown in the appendix, the magnetic superpotential requires an additional term, and is thus

$$W_{\mathrm{magn}} = \frac{h}{\mu^2}\sum_i M_i \overline{q} Y^{k-i} q - \left(\frac{\det M^{(1)}}{\Lambda^{(1)\;3N/2-(F+1)}} + \frac{\det M^{(2)}}{\Lambda^{(2)\;3N/2-(F+1)}}\right), \tag{2.7}$$

where

$$M^{(1),(2)} = \frac{M_1 \pm v M_2}{2v}, \tag{2.8}$$

and, consistently with the general case, $v=\sqrt{\frac{1}{2}\,\mathrm{Tr}\,Y^2}$.

At the origin $v=0$, the adjoint doesn't decouple from the low-energy theory. and the moduli space can be described at all energies by generalized mesons

$$(M_j)^i_{\bar{i}} = \overline{Q}_{\bar{i}} X^{j-1} Q^i\,; \qquad j=1,\ldots,k\,; \quad i,\bar{i}=1,\ldots,F\,, \tag{2.9}$$

baryons

$$B^{(n_1,n_2,\dots,n_k)} = Q^{n_1}(XQ)^{n_2}\cdots(X^{k-1}Q)^{n_k} \tag{2.10}$$

and, finally, $\mathrm{Tr}\,X^j$ with $j = 1,\dots,k$. The mesons (2.9) can be also thought of as blocks of the matrix

$$\begin{pmatrix} \overline{Q}Q & \overline{Q}XQ & \dots & \overline{Q}X^{k-2}Q & \overline{Q}X^{k-1}Q \\ \overline{Q}XQ & \overline{Q}X^2Q & \dots & \overline{Q}X^{k-1}Q & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ \overline{Q}X^{k-1}Q & 0 & \dots & 0 & 0 \end{pmatrix} \tag{2.11}$$

constructed from the "dressed" quarks and anti-quarks

$$Q_{(l)} = X^{(l-1)}Q\,; \qquad \overline{Q}_{(l)} = X^{(l-1)}\overline{Q}\,; \qquad l = 1,\dots k\,. \tag{2.12}$$

A mapping between the above gauge-invariant operators and those of the magnetic dual can be established as follows: the mesons in eq. (2.9) correspond to the elementary singlets of the magnetic theory. The correspondence between the electric baryons (B) defined in eq. (2.10) and magnetic ones (b), constructed in an analogous way out of q and Y, is

$$B^{(n_1,n_2,\dots,n_k)} \leftrightarrow b^{(m_1,m_2,\dots,m_k)}\,, \qquad m_l = F - n_{k+1-l}\,, \qquad l = 1,\dots,k \tag{2.13}$$

and the traces $\mathrm{Tr}\,X^j$ are simply mapped to the analogous $-\mathrm{Tr}\,Y^j$.

Non-trivial tests of the duality conjecture include:

- The fact that the charge assignment of table 1 and table 2, necessary for 't Hooft anomaly matching, is also compatible with the above mapping.
- The fact that the moduli space of the electric and magnetic theory are equivalent after including instanton effects as we show in the appendix.
- The fact that the duality is preserved under mass deformations.

The discussion in refs [2, 3, 4] established a picture of the IR behavior of the theory at $v = 0$ for all the values of F such that $kF - N > 1$: the theory is in the free electric phase for $F > 2N$, in the free magnetic case for $F < \frac{2}{2k-1}N$ and in the non-abelian Coulomb phase for the values of F in the intermediate range.

Furthermore, Csáki and one of the authors (HM) studied the case $kF - N = 1$ [7], by adding a mass deformation to the case $kF - N = k + 1$. They found that for $kF - N = 1$ the theory is always confining, and obtained explicitly the confining superpotential as a k-instanton effect in the magnetic theory.

Along the same line, we want to investigate whether duality can elucidate the behavior of the theory for $kF = N$. In section 3 we will analyze in detail the electric theory and obtain the quantum modification of the moduli space in the limit $v \to 0$. This quantum modification will be obtained explicitly as an instanton effect in section 4, in the framework of the dual magnetic theory.

3. The Electric theory with $kF = N$

Here we investigate the $\mathrm{SU}(kF)$ theory with F pairs of quarks Q, $\overline{Q}$ and an adjoint field X with the superpotential

$$W = \frac{h}{k+1}\,\mathrm{Tr}\,X^{k+1}\,. \tag{3.1}$$

The classical moduli space is given in terms of the mesons $M_j = \overline{Q}X^{j-1}Q$ ($j = 1, 2, \ldots, k$), baryons $B = Q^F(XQ)^F \cdots (X^{k-1}Q)^F$, $\overline{B} = \overline{Q}^F(X\overline{Q})^F \cdots (X^{k-1}\overline{Q})^F$, $\mathrm{Tr}\,X^j$, $j = 2, 3, \ldots, k$. If we define the matrix $\mathcal{M}$ such that $\mathcal{M}_{ij} = M_{i+j-1}$ for $i + j \leq k + 1$ and $\mathcal{M}_{ij} = v^k M_{i+j-(k+1)}$ otherwise, baryons and mesons are subject to the constraint

$$\det \mathcal{M} - B\overline{B} = 0\,. \tag{3.2}$$

In the limit $v \to 0$ M reduces to the matrix in eq. (2.11) and the constraint (3.2) simplifies to

$$(-1)^{\frac{k(k-1)}{2}} \det(M_k)^k - B\overline{B} = 0\,. \tag{3.3}$$

Let us briefly discuss the number of degrees of freedom of the classical moduli space. We have k M_j's each with F^2 components, B and $\overline{B}$, subject to the constraint (3.2), plus the traces $\mathrm{Tr}\,X^j$, $j = 2, 3, \ldots, k$. Therefore, we have $kF^2 + k$ dimensions.

Along the F-flat and D-flat direction $X^k = v^k I$ with $v \neq 0$, X takes the form

$$X = v\,\mathrm{diag}(\underbrace{1, \ldots, 1}_{n_1}, \underbrace{\omega, \ldots, \omega}_{n_2}, \ldots, \underbrace{\omega^{k-1}, \ldots \omega^{k-1}}_{n_k})$$

with $\omega = e^{2\pi i/k}$. It is easy to see, however, that only the choice $n_1 = \cdots = n_k = F$ is left on the quantum moduli space. For any other choice at least one of the remaining $\mathrm{SU}(n_j)$ gauge groups satisfies $n_j > F$, and hence a dynamical superpotential is generated and the moduli space is lifted quantum mechanically. For the only possible choice $n_1 = \cdots n_k = F$, the gauge group is broken to $\mathrm{SU}(F)^k \times \mathrm{U}(1)^{k-1}$, where each $\mathrm{SU}(F)$ factor has F flavors. The dynamical scale $\Lambda^{(j)}$ of the low-energy j-th $\mathrm{SU}(F)$ theory is given by

$$\Lambda^{(j)2F} = \Lambda^{2kF-F}\frac{(hk(v\omega^{j-1})^{k-1})^F}{(k(v\omega^{j-1})^{(k-1)})^{2F}} = \frac{\Lambda^{2kF-F}h^F}{k^F(v\omega^{j-1})^{(k-1)F}} \tag{3.4}$$

where the combination hv^{k-1} is the mass of the adjoint chiral multiplet due to the superpotential coupling. Here and hereafter, we do not keep prefactors that depend on $O(1)$ numbers and k, but retain only power dependencies. Note that the dynamical scales of $\mathrm{SU}(F)$ factors are different in phase. We have the quantum modified moduli spaces for each $\mathrm{SU}(F)$ factor:

$$\begin{aligned} W &= \sum_{j=1}^{k} X_j\left(\det M^{(j)} - B^{(j)}\overline{B}^{(j)} - \Lambda^{(j)2F}\right) \\ &= \sum_{j=1}^{k} X_j\left(\det M^{(j)} - B^{(j)}\overline{B}^{(j)} - \frac{h^F\Lambda^{2kF-F}}{k^F(v\omega^{j-1})^{(k-1)F}}\right), \end{aligned} \tag{3.5}$$

where X_j are Lagrange multiplier fields. The baryons $B^{(j)}$ and $\overline{B}^{(j)}$ are charged under the unbroken $\mathrm{U}(1)^{k-1}$ gauge groups. The photons correspond to the gauge-invariants $\mathrm{Tr}\, X^j W_\alpha$ for $j = 1, \ldots, k-1$. There are kF^2 meson degrees of freedom and $2k$ baryons, subject to k constraints. Naively, this leaves $kF^2 + k$ degrees of freedom, while, according to the previous counting, at each point in the $k-1$-dimensional space $\mathrm{Tr}\, X^l$, $l = 2, \ldots, k$ there should be $kF^2 + 1$. However, this is not a contradiction because of $\mathrm{U}(1)^{k-1}$ gauge factors. On a generic point on the moduli space, the baryon fields break the $\mathrm{U}(1)^{k-1}$ gauge group completely. Therefore, $k-1$ of them are "eaten" by the Higgs mechanism. The number of chiral superfields in the low-energy limit is therefore still $kF^2 + 1$.

Now the challenge is to study the limit $v \to 0$. Clearly, the moduli space described by eq. (3.5) is singular as $v \to 0$. We approach this limit in two different ways.

The first method is to approach $v \to 0$ when the $\mathrm{U}(1)^{k-1}$ factors are always broken. It is then possible to explicitly integrate out unnecessary degrees of freedom from eq. (3.5). The point is that $k-1$ degrees of freedom out of baryons are "eaten" and hence the remaining degrees of freedom can be parameterized by

$$N^{(i)} \equiv B^{(i)}\overline{B}^{(i)}\,, \qquad \mathcal{B} \equiv \prod_i B^{(i)}\,, \qquad \overline{\mathcal{B}} \equiv \prod_i \overline{B}^{(i)}\,, \tag{3.6}$$

subject to a constraint

$$\mathcal{B}\overline{\mathcal{B}} = \prod_i N^{(i)}\,. \tag{3.7}$$

And the l.h.s. of the constraint is related to the original baryon operators by

$$B\overline{B} = k^{kF} v^{k(k-1)F} (-\omega)^{\frac{k(k-1)F}{2}} \mathcal{B}\overline{\mathcal{B}}\,. \tag{3.8}$$

The latter relation is obtained in the following way. The original baryon operator is defined as the determinant of the matrix

$$\begin{pmatrix} Q^{(1)} & Q^{(2)} & \ldots & Q^{(k)} \\ (XQ)^{(1)} & (XQ)^{(2)} & \ldots & (XQ)^{(k)} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ (X^{k-1}Q)^{(1)} & (X^{k-1}Q)^{(2)} & \ldots & (X^{k-1}Q)^{(k)} \end{pmatrix} \tag{3.9}$$

where each of the entries is an $F \times F$ matrix and the upper index refers to the group of colors associated with the eigenvalue $v\omega^i$ of X. If we define the matrix Ω such that $\Omega_{ij} = \mathcal{I}_{F\times F}\ \omega^{(i-1)(j-1)}$ where $\mathcal{I}$ is the identity matrix, the determinant of the matrix in eq. (3.9) is $v^{k(k-1)F/2}$ $(\det \Omega)$ $\mathcal{B}$ with $\det \Omega = k^{kF/2}\ (-\omega)^{\frac{k(k-1)F}{4}}$.

The $N^{(i)}$ are given directly from eq. (3.5) and we find

$$\begin{aligned} B\overline{B} &= k^{kF} v^{k(k-1)F} (-\omega)^{\frac{k(k-1)F}{2}} \prod_j \left(\det M^{(j)} - \frac{h^F \Lambda^{2kF-F}}{k^F (v\omega^{j-1})^{(k-1)F}} \right) \\ &= \prod_j \left(k^F v^{(k-1)F} (-\omega)^{\frac{(k-1)F}{2}} \det M^{(j)} - (-)^{\frac{(k-1)F}{2}} h^F \Lambda^{2kF-F} \right). \end{aligned} \tag{3.10}$$

Now by rewriting $M^{(j)}$ as

$$M^{(j)} = \frac{1}{k}\sum_{l=1}^{k} \frac{M_l}{v^{l-1}} \omega^{-(j-1)(l-1)}\,, \tag{3.11}$$

we can smoothly take the limit

$$v^{(k-1)} M^{(j)} \to M_k \omega^{-(j-1)(k-1)} = \frac{M_k}{k}\omega^{j-1}\,. \tag{3.12}$$

Then the constraint eq. (3.10) becomes

$$B\overline{B} = \prod_j (-1)^{\frac{(k-1)F}{2}} \left(\det M_k - h^F \Lambda^{2kF-F}\right). \tag{3.13}$$

This is the quantum modified constraint among composites at $v \to 0$, *as long as* you approach the origin with all the gauge groups always completely broken. Note that the Higgs phase and confining phase are equivalent in this theory because of quarks in the fundamental representation.

What about taking the limit $v \to 0$ with keeping some or all of U(1)'s unbroken? There are many reasons to believe that this limit leads to an interacting superconformal theory. One way to see it is as follows. We can always force the baryons to vanish in eq. (3.5), by adding a mass term to the quarks. By adding a common mass term for simplicity,

$$W = \sum_{j=1}^{k} X_j \left(\det M^{(j)} - B^{(j)}\overline{B}^{(j)} - \frac{h^F \Lambda^{2kF-F}}{k^F (v\omega^{j-1})^{(k-1)F}}\right) + m \operatorname{Tr} M_1\,, \tag{3.14}$$

and noting

$$M_1 = \sum_{j=1}^{k} M^{(j)}\,, \tag{3.15}$$

we can solve $\partial W/\partial M^{(j)} = 0$ to find that $X_j \neq 0$. This is enough to force all baryons to vanish. Then we can ask the question what happens in the $v \to 0$ limit. Because quarks are massive, we can integrate them out first instead, and add the superpotential $h \operatorname{Tr} X^{k+1}$ afterwards. Once the quarks are integrated out, the theory is nothing but the $N = 2$ Yang-Mills theory, whose curve is known. Adding the superpotential is known to make the theory flow to an Argyres-Douglas fixed point. It was worked out explicitly for the SU(3) and $k = 2$ case [8], but it is believed that any SU(N) theory with any k would lead to such non-trivial fixed-point theories, as long as $k < N$. Therefore for $F \geq 2$, the theory will flow to superconformal theories. When $F = 1$, however, the superpotential is (presumably) irrelevant, and the theory is given by the Coulomb branch of the entire SU(N) $N = 2$ Yang-Mills.

4. The dual magnetic theory with $kF = N$

For $F = kN$, the general strategy to study the magnetic theory is to start from the SU(k) theory with $F + 1$ flavors, where the spectrum is given by the fields in table 2, and then add a deformation

$$W_{def} = m(M_1)_{F+1,F+1} \tag{4.1}$$

corresponding to the mass term $mQ_{F+1}\overline{Q}_{F+1}$ of the electric theory.

JHEP05(2002)055

We will begin by discussing a generic k case, and then we will consider the special $k = 2$ case, that presents some special subtleties.

4.1 $k \neq 2$

In the generic case $k \neq 2$ the superpotential is

$$W_{\rm magn} = -\frac{h}{k+1}\,\mathrm{Tr}\,Y^{k+1} + \frac{h}{\mu^2}(M_1 \overline{q} Y^{k-1} q + \cdots + M_k \overline{q} q)\,. \tag{4.2}$$

Pretty much like in the electric theory, on the moduli space Y^k must be proportional to the identity. In the following, we will analyze first the flat direction $Y^k \neq 0$ and then consider the special point $Y^k = 0$.

4.1.1 $v^k = \frac{1}{k}\,\mathrm{Tr}\,Y^k (= \frac{1}{N}\,\mathrm{Tr}\,X^k) \neq 0$

In the points of the moduli space with $v^k = \frac{1}{2}\,\mathrm{Tr}\,Y^k \neq 0$ the adjoint of the magnetic theory (like that of the electric theory) is diagonalizable with a complex gauge transformation. In the electric theory with $F+1$ flavors, the only vacuum which is stable under a deformation such as a mass term for the $F+1$-th flavor breaks $\mathrm{SU}(N)$ to $\mathrm{SU}(N/k) \times \mathrm{U}(1)^{k-1}$, and correspondingly in the magnetic theory $\mathrm{SU}(k)$ is broken to $\mathrm{U}(1)^{k-1}$. The baryons in the k decoupled sectors of the magnetic theory are

$$b_i^{(j)} = \frac{\Lambda^{(j)\,(F-1/2)} h^{\frac{F+1}{k}}}{(\mu^{(j)})^{1/2}} q_i^{(j)}\,, \qquad i = 1, F+1\,, \quad j = 1, k\,, \tag{4.3}$$

where the $\Lambda^{(j)}$'s are the dynamical scales of the k (electric) subsectors see eq. (3.4)and the $\mu^{(j)}$'s are related to the original scale μ by the relation

$$\mu^{(i)} = \frac{\mu^2}{k(v\omega^{j-1})^{k-1} h}\,. \tag{4.4}$$

The latter is obtained by matching the relation $\Lambda_{el}^{2kF-F} \Lambda_{\rm magn}^{-F} = (\mu/h)^{2F}$ of the high energy theory to the relation $(\Lambda_{el}^{(i)})^{2F} (\Lambda_{\rm magn}^{(i)})^{-F} = (\mu^{(i)})^F$ of the low energy sectors.

The mesons $M^{(j)}$ of the subsectors are

$$M^{(j)} = \frac{1}{k} \sum_{l=1}^{k} \frac{M_l}{v^{l-1}} \omega^{-(j-1)(l-1)}\,. \tag{4.5}$$

In each of the k decoupled sector there are F^2 mesons, one baryon and one anti-baryon subject to one constraint (which classically doesn't involve the mesons). At a generic point of the moduli space, the baryons break the residual $\mathrm{U}(1)$ gauge factors, and the total number of degrees of freedom is $kF^2 + 1$.

After decoupling the Y degrees of freedom, the tree level superpotential is:

$$W = \sum_{j=1,k} M^{(j)} \frac{\overline{q}^{(j)} q^{(j)}}{\mu^{(j)}}\,. \tag{4.6}$$

However, as the non-abelian factor of the magnetic gauge group is completely broken, instanton effects have to be added to the above superpotential as matching conditions. After instanton effects are included, we expect the superpotential to have the same form as that of the electric theory with $F+1$ flavors (i.e. $\left[\frac{BMB}{\Lambda^{N-1}} - \det M\right]$ for each of the decoupled sectors), so that after massive fields are integrated out the algebraic relations between baryons and mesons are modified in such a way as to reproduce those of eq. (2.5).

The required instanton term is

$$\sum_j -\frac{\det M^{(j)}}{\Lambda^{(j)\ (2F-1)}} \,. \tag{4.7}$$

If we now add the deformation (4.1), which, in terms of the mesons of the decoupled sectors, reads

$$m \sum_{j=1,k} M^{(j)} \,, \tag{4.8}$$

the fields $M^{(1),\ldots,()}_{F+1,i}$, $M^{(1),\ldots,(k)}_{i,F+1}$, $i = 1, F+1$, $q^{(1),\ldots,(k)}_j$, $\overline{q}^{(1),\ldots,(k)}_j$, $j = 1, F$ become massive and can be integrated out. We can then identify

$$b^{(i)} \equiv \frac{\Lambda^{(1)\,F-1/2}}{\mu^{(1)\,1/2}} q^{(i)}_{F+1} \,, \tag{4.9}$$

i.e. the operators $b^{(i)}_{F+1}$ in eq. (4.3), and the analogous operators with $\overline{q}^{(i)}_{F+1}$, with the baryons and anti-baryons of the theories with zero colors, and the constraints between mesons and baryons turn out to be exactly like those of the electric theory:

$$\det M^{(i)} - b^{(i)}\overline{b}^{(i)} = \tilde{\Lambda}^{(i)} \tag{4.10}$$

4.1.2 $\operatorname{Tr} X^k = \operatorname{Tr} Y^k = 0$

In this case on the classical moduli space we have

$$\overline{q} = (u', 0, \ldots, 0) \,, \qquad q = (0, 0, \ldots, 0, u''), Y_{i,j} = \delta_{i+1,j} u_i \qquad i, j = 1, k \tag{4.11}$$

with $u'u''\Pi_{j=1,k} u_j = -m\mu^2$. Notice that, being the u's further constrained by $k-1$ nontrivial D-flatness conditions, only one of the vevs is left unconstrained. Adding the latter to the kF^2 mesons we have kF^2+1 degrees of freedom, the same number as in the electric theory for $\operatorname{Tr} X^j = 0$.

The instanton generated superpotential (4.7) reduces to

$$\begin{aligned} W_{\text{inst}} = & -\frac{1}{kh^F \Lambda^{2kF-(F+1)}} \times \\ & \times \sum_{l,m,\ldots,z} \delta_{l(k-l)+m(k-m)+\cdots+z(k-z),k-1} \epsilon_{j_1,j_2,\ldots j_{F+1}} (M_l)_{1,j_1} (M_m)_{2,j_2} \cdots (M_z)_{F+1,j_{F+1}} \,. \end{aligned} \tag{4.12}$$

In the (high-energy) theory with $F+1$ flavors, the F-flatness conditions obtained by adding the new superpotential to that of eq. (2.6) reproduce the (compositeness) constraints between electric baryons and mesons. Notice that the superpotential (4.12) contains in particular the term

$$-\frac{M_1 \mathrm{cof}(M_k)}{h^F \Lambda^{2kF-(F+1)}} \tag{4.13}$$

which is crucial to obtain the quantum modified constraint in the low energy theory.

Indeed, by integrating out the massive fields we find the quantum generated constraint

$$\langle \det M_k \rangle - \frac{\Lambda^{2N-(F+1)} h^{F+1}}{\mu^2} \langle \overline{q}_{F+1} Y^{k-1} q_{F+1} \rangle = h^F \Lambda^{2N-(F+1)} \times m = h^F \tilde{\Lambda}^{(2N-F)}, \tag{4.14}$$

where $\tilde{\Lambda}$ is the scale of the low energy theory with F flavors.

Being (for $v = 0$) $\mathcal{M}$ of the form

$$\begin{pmatrix} M_1 & M_2 & \dots & M_{k-1} & M_k \\ M_2 & M_3 & \dots & M_k & 0 \\ . & & & & \\ . & & & & \\ . & & & & \\ M_k & 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.15}$$

we find that it is

$$(-\det \mathcal{M})^{1/k} = \det M_k\,. \tag{4.16}$$

Moreover, in the low energy theory with zero colors the product of baryon and anti-baryon can be identified with

$$b\overline{b} = (-)^{\frac{k(k-1)F}{2}} \frac{\tilde{\Lambda}^{k(2N-F)}}{\mu^{2k}} h^{k(F+1)} (\overline{q}_{F+1} Y^{k-1} q_{F+1})^k\,. \tag{4.17}$$

This can be understood in the following way. Consider the magnetic ($\mathrm{SU}(k)$) theory with $F+1$ flavors. Among the $\binom{k(F+1)}{k}$ baryons of the magnetic theory there is

$$(-)^{\frac{F+1}{2}} \frac{\Lambda^{2N-(F+1)} h^{F+1}}{\mu^2} q_{F+1} (Y q_{F+1}) \cdots (Y^{k-1} q_{F+1}) \tag{4.18}$$

(for the numerical factor see ref. [4], eq. (2.20)). According to the duality vocabulary, this corresponds to the baryon

$$Q_1 \cdots Q_F (XQ_1) \cdots (XQ_F) \cdots (X^{k-1}Q_1) \cdots (X^{k-1}Q_F) \tag{4.19}$$

of the electric theory. After integrating out the $F+1$-th flavor, the ¡(opposite of the) latter is the only baryon remaining in the electric theory, and, if the duality properties are preserved under mass deformation, this should still correspond to the (opposite of the) operator (4.18) in the magnetic theory.

Substituting eqs. (4.16) and (4.17) into eq. (4.14), we can rewrite the quantum modified constraint as

$$(-)^{\frac{k(k-1)F}{2}} \langle (\det \mathcal{M}) \rangle = (h^F \tilde{\Lambda}^{2N-F} (-)^{\frac{(k-1)F}{2}} \langle (b\overline{b})^{1/k} \rangle)^k \tag{4.20}$$

which is in complete agreement with the quantum modified constraint on the electric theory.

The constraint (4.20) is also satisfied when $q = \overline{q} = Y = 0$ and only the mesons get a vev, in which case the SU(k) gauge symmetry is unbroken and much of the above discussion seems not to hold. Indeed, it is not even obvious from the above considerations that this point belongs to the moduli space. The instanton superpotential is generated if the gauge group is broken; if it was not, the F-flatness condition would be the classical one, which is not satisfied by $q = \overline{q} = Y = 0$. On the other hand this point can be reached from the direction $b^{(i)} = \overline{b^{(i)}} = 0$ in the limit $v \to 0$ and from the direction $v = 0$, $\det M_k = h^F \Lambda^{2N-F}$, $B(\overline{B}) = 0$, $\overline{B}(B) \to 0$ on the moduli space. In this limit, $\mathrm{U}(1)^{k-1}$ gauge invariance is unbroken and additional charged massless fields can arise at singularities on the moduli space where SU(k) is recovered classically. Therefore we conclude that this limit is on the moduli space, where the theory becomes superconformal.

4.1.3 Explicit derivation of the instanton term

The explicit derivation of the instanton term is very similar to the case $k = 2$. The 't Hooft effective vertex is given by

$$\Lambda_{\mathrm{magn}}^{2k-(F+1)} \tilde{q}^{F+1} \tilde{\overline{q}}^{F+1} \tilde{Y}^{2k} \lambda^{2k} \,. \tag{4.21}$$

In order to saturate the fundamental fermions, we use thus the following couplings:

- $\frac{h}{\mu^2} M_l \tilde{\overline{q}} Y^{k-l} \tilde{q}$, $\frac{h}{\mu^2} M_m \tilde{\overline{q}} Y^{k-m} \tilde{q}$,... for a total of $(F-1)$ times
- $\frac{h}{\mu^2} \tilde{M}_r \overline{q} Y^{k-r} \tilde{q}$ once
- $\frac{h}{\mu^2} \tilde{M}_s \tilde{\overline{q}} Y^{k-s} q$ once

in such a way that the total power of Y is $k-1$, to obtain

$$\Lambda_{\mathrm{magn}}^{2k-(F+1)} M_l \cdots \tilde{M}_r \tilde{M}_s \tilde{q}^2 \tilde{\overline{q}}^2 \tilde{Y}^{2k} \lambda^{2k} Y^{k-1} q \overline{q} \frac{h^{F+1}}{\mu^{2(F+1)}} \,. \tag{4.22}$$

Then we use:

- $q^* \lambda \tilde{q}$ once
- $\overline{q}^* \lambda \tilde{\overline{q}}$ once
- $Y^* \lambda \tilde{Y}$ (2k-2) times
- $-h Y^{k-1} \tilde{Y}^2$ once

to end up with

$$\Lambda_{\mathrm{magn}}^{2k-(F+1)} M_l \cdots \tilde{M}_r \tilde{M}_s \frac{h^{F+2}}{\mu^{2(F+1)}} = \frac{1}{h^F \Lambda^{(2k-1)F-1}} M_l \cdots \tilde{M}_r \tilde{M}_s \tag{4.23}$$

(where we used the relationship $\Lambda^{(2K-1)F-1} \Lambda_{\mathrm{magn}}^{(2K-1)-F} = (\mu/h)^{2F+2}$) which term can be embedded in the superpotential term (4.12). These are the only terms compatible with all the symmetries, in particular the $\mathrm{U}(1)_R$ symmetry. When $\mathrm{Tr}\, Y^k \neq 0$ this superpotential turns into the k terms in eq. (4.7). In this case there are no symmetry reasons to forbid terms which mix different $\hat{M}_i$'s. The only reason is dynamical: as the k sectors are completely decoupled for $v \neq 0$, no term which mixes them can be generated.

4.2 $k = 2$

As $k \neq 2$ case, we add one extra flavor to obtain the qmagnetic SU(2) theory, and decouple the extra flavor to find the quantum modified moduli space. However, the magnetic SU(2) theory needs to be augmented by an additional term in the superpotential to obtain the same moduli spaces between the electric and magnetic theories as shown in the appendix.

The superpotential of the SU(2) theory is that of eq. (2.7), to which we add a mass deformation for the $F+1-$th flavor:

$$W_{\mathrm{magn}} = \frac{h}{\mu^2}(M_1 \overline{q} Y q + M_2 \overline{q} q) - \left(\frac{\det M^{(1)}}{\Lambda^{(1)\ 3N/2-(F+1)}} + \frac{\det M^{(2)}}{\Lambda^{(2)\ 3N/2-(F+1)}} \right) + m(M_1)_{F+1,F+1}\,. \tag{4.24}$$

Apart from the special choice of the superpotential, this is the theory considered in [10].

With a real gauge transformation, we can put q_{F+1} in the form

$$q_{F+1} = \begin{pmatrix} 0 \\ u_2 \end{pmatrix}. \tag{4.25}$$

With this gauge choice, in order to satisfy both F-flatness and D-flatness conditions, it must be

$$\overline{q}_{F+1} = (u_1, 0)\,, \qquad Y = \begin{pmatrix} 0 & u_3 \\ u_4 & 0 \end{pmatrix}, \tag{4.26}$$

with $u_1 u_2 u_3 = -m\mu^2$ (classically) and $D_3 = \frac{u_1^2}{2} + \frac{u_2^2}{2} - (u_3^2 + u_4^2) = 0$. This leaves two out of the four vevs (u_1, u_2, u_3, u_4) unconstrained. We also notice that after Y and q_{F+1} get vev, the combinations $q_i(M_1)_{i,F+1}$, $i = 1, \ldots, F$ and $q_i(M_2)_{i,F+1}$, $i = 1, \ldots, F+1$, become massive, and can be integrated out. The number of degrees of freedom is thus $2F^2 + 2$, equal to that of the electric theory.

In particular u_4 can be either vanishing or not. In case it is not, it is $\mathrm{Tr}\, Y^2 \neq 0$, Y can be put (with a complex gauge transformation) in diagonal form and the behavior of the theory has to reproduce that of the electric theory for $v \neq 0$. We will analyze this case first, and then discuss the case $u_4 = 0$ as the limit for $\mathrm{Tr}\, Y^2 \to 0$.

4.2.1 $v^2 = \frac{1}{2}\ \mathrm{Tr}\, Y^2 (= \frac{1}{N}\ \mathrm{Tr}\, X^2) \ \neq 0$

When $u_4 \neq 0$, Y can be put in the diagonal form

$$Y = \begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix}, \tag{4.27}$$

with $v = \sqrt{u_3 u_4}$, with a complex gauge transformation.

The theory is split in two decoupled subsectors and the superpotential can be written as:

$$\begin{aligned} W_{\mathrm{mag}\,1} = {} & \left(\frac{1}{\mu^{(1)}} \overline{q}^{(1)} q^{(1)} M^{(1)} + \frac{1}{\mu^{(2)}} \overline{q}^{(2)} q^{(2)} M^{(2)} \right) - \\ & - \left(\frac{\det M^{(1)}}{\Lambda^{(1)\ 3N/2-(F+1)}} + \frac{\det M^{(2)}}{\Lambda^{(2)\ 3N/2-(F+1)}} \right) + m(M^{(1)}_{F+1,F+1} + M^{(2)}_{F+1,F+1})\,, \end{aligned} \tag{4.28}$$

where $\mu^{(1),(2)} = \pm \frac{\mu^2}{2vh}$, consistently with the general formula (4.4).

$\hat{M}_{1,2}$ can be identified with the mesons of the two decoupled sectors, while the baryons are

$$\frac{\Lambda^{(1)\ F-1/2}}{\mu^{(1)\ 1/2}} q_i^{(1)}\,, \frac{\Lambda^{(2)\ F-1/2}}{\mu^{(2)\ 1/2}} q_i^{(2)}\,, \tag{4.29}$$

where $\Lambda^{(1),(2)}$ are the dynamical scales of the two *electric* subsectors and the anti-baryons the analogous operators with $\overline{q}$.

In each of the two decoupled sector there are F^2 mesons, one baryon and one anti-baryon subject to one constraint (which classically doesn't involve the mesons). At a generic point of the moduli space, the baryons break the residual U(1) gauge group, and the total number of degrees of freedom is $2F+1$.

The magnetic gauge group is broken to U(1), and we might expect that instanton effects had to be added, but it turns out that in the absence of the superpotential term involving $\operatorname{Tr} Y^{k+1}$ such effects vanish.

After massive fields ($M^{(1),(2)}_{F+1,i}$, $M^{(1),(2)}_{i,F+1}$, $i = 1, F+1$, $q_j^{(1),(2)}$, $\overline{q}_j^{(1),(2)}$, $j = 1, F$) are integrated out, we can identify

$$b^{(1)} \equiv \frac{\Lambda^{(1)\ F-1/2}}{\mu^{(1)\ 1/2}} q_{F+1}^{(1)}\,, \qquad b^{(2)} \equiv \frac{\Lambda^{(2)\ F-1/2}}{\mu^{(2)\ 1/2}} q_{F+1}^{(2)} \tag{4.30}$$

and the analogous operators with $\overline{q}_{F+1}$ with the baryons and anti-baryons of the theories with zero colors and the constraints between mesons and baryons turn out to be exactly like those of the electric theory:

$$\det M^{(1),(2)} - b^{(1),(2)}\overline{b}^{(1),(2)} = \tilde{\Lambda}^{(1),(2)}\,, \tag{4.31}$$

where $\tilde{\Lambda}^{(1),(2)}$ are the scales of the low energy theories (with F flavors) for the two decoupled sectors, related to the high energy scales by $\tilde{\Lambda}^{(1),(2)\,2F} = m\Lambda^{(1),(2)\,(2F-1)}$.

4.2.2 $\operatorname{Tr} Y^2 (= \operatorname{Tr} X^2) = 0$

In the limit $v = 0$, the additional superpotential term in (2.7) reduces to

$$W_{\text{add}} = -\left(\frac{M_1 \operatorname{cof}(M_2)}{h^F \Lambda^{2N-(F+1)}}\right). \tag{4.32}$$

In the presence of this term, the low energy theory, containing only the (now $F \times F$) meson matrices M_1 and M_2 and another degree of freedom out of the parameters (u_1, u_2, u_3) of eqs.(4.25),(4.26), is characterized by the (quantum generated) constraint

$$\langle \det M_2 \rangle - \frac{\Lambda^{2N-(F+1)} h^{F+1}}{\mu^2} \langle \overline{q}_{F+1} Y q_{F+1} \rangle = \Lambda^{2N-(F+1)} m h^F = h^F \tilde{\Lambda}^{(2N-F)}\,, \tag{4.33}$$

where $\tilde{\Lambda}^{(2N-F)} = m\Lambda^{(2N-(F+1))}$ is the scale of the low energy theory with F flavors.

Indeed, in the low-energy theory, the operator corresponding to the product of the baryon and the anti-baryon in the electric theory is

$$b\overline{b} \equiv (-)^F \frac{\Lambda^{2(3F-1)} h^{2(F+1)}}{\mu^4} (\overline{q}_{F+1} Y q_{F+1})^2 = (-)^F \frac{\tilde{\Lambda}^{6F} h^{2(F+1)}}{m^2 \mu^4} (\overline{q}_{F+1} Y q_{F+1})^2\,. \tag{4.34}$$

which is the specialization to the case $k = 2$ of eq. (4.17).

Therefore, the moduli space for $N = 2F$ is characterized by the constraint

$$\langle (-)^F \det \mathcal{M} \rangle = \left(h^F \tilde{\Lambda}^{(2N-F)} + (-)^F \sqrt{\langle b\bar{b} \rangle} \right)^2 . \tag{4.35}$$

and this agrees with the quantum modified constraint on the electric moduli space.

At the singular point $M_2 = h^F \tilde{\Lambda}^{2N-F}$, $b = \bar{b} = 0$, where the SU(2) gauge group is unbroken. Dynamics there must be superconformal, which fact is also consistent with the suggestion of ref. [10].

5. Conclusion

In this paper, we have investigated the dynamics of supersymmetric gauge theories with less flavor than the Kutasov-duals. By integrating out a flavor from the known duality pair, we demonstrated that the theory has a quantum-modified moduli space, as expected from the analogy to the Seiberg duality in SU(N) QCD. However, a point on the moduli space becomes superconformal, a distinct behavior from the Seiberg duality. In fact, such a behavior had been seen in an isolated example of SU(2) gauge theory with two doublets and a triplet, and our result generalizes this behavior to a whole class of theories. We also pointed out that in the case $k = 2$, $N = 2F - 2$, the magnetic theory is not capable to reproduce the structure of the moduli space in the electric theory unless we supply the magnetic superpotential with an additional term, that was neglected in previous literature.

This result gives additional information on the phase space of these theories.

A. Non-trivial Checks of Kutasov duality

In this section we present additional new non-trivial checks of Kutasov duality, showing in particular that in the case $N = 2F - 2$ the additional superpotential term in (2.7) has indeed to be added in the magnetic theory. The question is if moduli spaces agree between the electric and magnetic theories. This check had not been done explicitly in literature to the best of our knowledge. We focus on the case $N = k(F - n)$, $n > 0$ because it is relevant to our discussion of the quantum modified moduli space when $N = kF$ $(n = 0)$.

In the electric theory, the dressed quarks form a $N \times kF$ matrix, and hence the dressed meson matrix $\mathcal{M}$ has a rank less than or equal to $N = k(F - n) < kF$. In this case, the following classical constraint holds

$$(\mathcal{M})^{j_1}_{i_1} (\mathcal{M})^{j_2}_{i_2} \cdots (\mathcal{M})^{j_{F-n}}_{i_{F-n}} \epsilon^{i_1 i_2 \cdots i_{F-n} i_{F-n+1} \cdots i_F} \epsilon^{j_1 j_2 \cdots j_{F-n} j_{F-n+1} \cdots j_F} = B^{i_{F-n+1} \cdots i_F} \bar{B}_{j_{F-n+1} \cdots j_F} . \tag{A.1}$$

Here, the indices run over kF values. In particular, when not along the flat direction $\operatorname{Tr} X^k \neq 0$, the only surviving piece in the left-hand side is given by M_k. Therefore,

$$\begin{aligned} &\left[(M_k)^{j_1}_{i_1} (M_k)^{j_2}_{i_2} \cdots (M_k)^{j_{F-n}}_{i_{F-n}} \epsilon^{i_1 i_2 \cdots i_{F-n} i_{F-n+1} \cdots i_F} \epsilon^{j_1 j_2 \cdots j_{F-n} j_{F-n+1} \cdots j_F} \right]^k = \\ &\quad = B^{i_{F-n+1} \cdots i_F; i_{F-n+1} \cdots i_F; \cdots i_{F-n+1} \cdots i_F} \bar{B}_{j_{F-n+1} \cdots j_F; j_{F-n+1} \cdots j_F; \cdots j_{F-n+1} \cdots j_F} . \end{aligned} \tag{A.2}$$

Here, the indices run only over the original flavors 1–F.

The same constraint is reproduced in the magnetic $\mathrm{SU}(kn)$ theory due to k-instanton effect. When rank$M_k = F - n$, we can integrate out $F - n$ dual quarks from the theory such that the low-energy dynamical scale is given by

$$\Lambda_{\mathrm{LE\,magn}}^{2kn-n} = \Lambda_{\mathrm{magn}}^{2nk-F} \widehat{\det} \left(\frac{h}{\mu^2} M_k \right) . \tag{A.3}$$

$\widehat{\det}$ is the non-vanishing minor determinant for the rank $F - n$ matrix. There is a unique baryon operator

$$b = q^k (Yq)^k \cdots (Y^{k-1}q)^k \tag{A.4}$$

in the low-energy $\mathrm{SU}(kn)$ theory with n flavors, and it breaks the gauge group completely. Then the instanton effects need to be considered. The k-instanton background gives k zero modes to each flavor, $2nk^2$ zero modes to both gaugino and $\tilde{Y}$. Along the baryon (and anti-baryon) direction, each quark zero modes combine with one gaugino zero mode to give the corresponding squark VEV; the remaining gaugino zero modes combine with $2nk(k-1)$ of the $\tilde{Y}$ ones to give $2nk(k-1)$ powers of Y; $nk(k-1)$ of the latter combine with the remaining $\tilde{Y}$ fermions giving a factor h^{nk}. Therefore, the instanton background gives the correlation function

$$(q_{F-n+1})^k \cdots (q_F)^k (\bar{q}_{F-n+1})^k \cdots (\bar{q}_F)^k Y^{nk(k-1)} = h^{nk} \Lambda_{\mathrm{LE\,magn}}^{k(2kn-n)} . \tag{A.5}$$

The left-hand side is nothing but $b\bar{b}$ for the remaining n flavors. We rewrite this result in terms of the electric scale using the matching condition

$$\Lambda^{2N-F} \Lambda_{\mathrm{magn}}^{2\tilde{N}-F} = \left(\frac{\mu}{h} \right)^{2F} . \tag{A.6}$$

We find

$$\begin{aligned} b\bar{b} &= h^{nk} \left(\frac{h}{\mu^2} \right)^{k(F-n)} (\widehat{\det} M_k)^k \frac{1}{\Lambda^{k(2N-F)}} \left(\frac{\mu}{h} \right)^{2kF} \\ &= \mu^{2kn} h^{-kF} \frac{1}{\Lambda^{k(2N-F)}} (\widehat{\det} M_k)^k . \end{aligned} \tag{A.7}$$

Matching between baryon operators is

$$B = h^{kF/2} \mu^{-\tilde{N}} \Lambda^{k(2N-F)/2} b \tag{A.8}$$

for appropriate flavor indices. Therefore, the factors in h, μ, and Λ all work out to give

$$B\bar{B} = (\widehat{\det} M_k)^k \tag{A.9}$$

for the relevant flavor combination.

Along the flat direction $v \neq 0$, the degrees of freedom of the adjoint field are integrated out, and in each of the k decoupled low-energy sectors we have some classical constraints between mesons and baryons analogous to those of eq. (A.1). If for every subsector it is $N^{(i)} < F + 1$, every subsector of the corresponding magnetic theory has a residual non-abelian gauge group, whose dynamics is known from refs [1] and has been found to

JHEP05(2002)055

reproduce the constraints of the electric theory. If for some subsectors it is either $N^{(i)} = F$ or $N^{(i)} = F{+}1$, in the corresponding magnetic subsector the (non-abelian part of the) gauge group is completely broken, and its superpotential must include instanton contributions from the theory with the adjoint. Take as an example the case $k = 4$, $n = 1$, and consider the flat direction where X has $F - 2$ eigenvalues v and $-v$ and F eigenvalues iv and $-iv$. In the magnetic theory we have two sectors (which we will label (3) and (4)) in which the gauge group is purely abelian. The one-instanton background gives 8 zero modes for gauginos and $\tilde{Y}$ and F zero modes for the squarks. Following the very same steps as in subsection 4.1.3, it is easy to see that the terms

$$-\left(\frac{\det M^{(3)}}{\Lambda^{(3)\ 2F}} + \frac{\det M^{(4)}}{\Lambda^{(4)\ 2F}}\right) \tag{A.10}$$

are generated.

But when the magnetic group is SU(2), $k = 2$, $n = 1$, there is no superpotential in the magnetic theory $\mathrm{Tr}\, Y^{k+1} = \mathrm{Tr}\, Y^3 = 0$. Because this superpotential term was crucial for the instanton effect to reproduce the classical constraint in the electric theory, the dual theory without the $\mathrm{Tr}\, Y^3$ term does not describe the same moduli space. The only way to remedy it is to introduce an additional term to the superpotential

$$W_{\mathrm{inst}} = -\left(\frac{\det M^{(1)}}{\Lambda^{(1)\ 2F}} + \frac{\det M^{(2)}}{\Lambda^{(2)\ 2F}}\right) \tag{A.11}$$

that reduces to

$$W_{\mathrm{inst}} = -\left(\frac{M_1 \mathrm{cof}(M_2)}{h^F \Lambda^{2N-F}}\right). \tag{A.12}$$

for $v = 0$.

As shown in section 4.2, this is sufficient to reproduce the constraints of the electric theory. Note that this additional term in the superpotential is a one-instanton contribution in the magnetic theory because we can rewrite it as

$$W_{\mathrm{inst}} = -\frac{h}{\mu^2} M_1 \mathrm{cof}\left(\frac{h}{\mu^2} M_2\right) \Lambda_{\mathrm{magn}}^{4-F}. \tag{A.13}$$

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